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SIMULTANEOUS LINEAR DIFFERENTIAL EQUATION

INTRODUCTION:

Here we consider the methods of solutions of differential equations involving more than two variables. The simplest form of such equations is that, in which the number of independent variables is one. The number of equations, which will connect these variables, will be equal to the number of dependent variables.

METHOD OF SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS:

There are two methods. We shall discuss the methods through examples. You can choose any one of these methods.

Example-1 :→ Solve $\frac{dx}{dt} - 7x + y = 0$

$\frac{dy}{dt} - 2x - 5y = 0$

Solution :→

METHOD-1 :→

The given equations can be expressed as

$$\begin{aligned} Dx - 7x + y &= 0 \\ Dy - 2x - 5y &= 0 \end{aligned} \quad , \text{ where } D \equiv \frac{d}{dt}.$$

i.e. $(D-7)x + y = 0 \longrightarrow (1)$

$-2x + (D-5)y = 0 \longrightarrow (2)$

Eliminating y between (1) and (2), we have

$$(D-5)(D-7)x + (D-5)y - [-2x + (D-5)y] = 0$$

$$\Rightarrow \{(D-5)(D-7) + 2\}x = 0$$

$$\Rightarrow (D^2 - 12D + 37)x = 0.$$

The auxiliary equation is $m^2 - 12m + 37 = 0$.

Now $m^2 - 12m + 37 = 0$

$$\Rightarrow m = \frac{12 \pm \sqrt{(12)^2 - 4 \times 37}}{2} = 6 \pm i$$

$\therefore x = e^{6t}(c_1 \cos t + c_2 \sin t)$, where c_1 and c_2 are arbitrary constants.

Then $\frac{dx}{dt} = 6e^{6t}(c_1 \cos t + c_2 \sin t)$
$\quad + e^{6t}(-c_1 \sin t + c_2 \cos t)$
$\quad = \{(6c_1 + c_2)\cos t + (6c_2 - c_1)\sin t\} e^{6t}$.

Putting the values of x and $\frac{dx}{dt} = Dx$ in ①, we have,

$$y = 7e^{6t}(c_1 \cos t + c_2 \sin t)$$
$$- \{(6c_1 + c_2)\cos t + (6c_2 - c_1)\sin t\} e^{6t}$$
$$= e^{6t}\{(7c_1 - 6c_1 - c_2)\cos t + (7c_2 - 6c_2 + c_1)\sin t\}$$
$$= e^{6t}\{(c_1 - c_2)\cos t + (c_1 + c_2)\sin t\}.$$

$\therefore$ The general solution of the given simultaneous linear differential equations is

$$x = e^{6t}\{c_1 \cos t + c_2 \sin t\}$$
and $$y = e^{6t}\{(c_1 - c_2)\cos t + (c_1 + c_2)\sin t\},$$ where c_1 and c_2 are arbitrary constants.

METHOD-2:→

The given equations are

$$\frac{dx}{dt} - 7x + y = 0 \longrightarrow ①$$
$$\frac{dy}{dt} - 2x - 5y = 0 \longrightarrow ②$$

Differentiating ① w.r.to t, we have

$$\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + \frac{dy}{dt} = 0 \longrightarrow ③$$

Now $\frac{dy}{dt} = 2x + 5y$ [From ②]
$= 2x + 5\left\{7x - \frac{dx}{dt}\right\}$ [From ①]
$= 37x - 5\frac{dx}{dt}$

Putting this value of $\frac{dy}{dt}$ in ③ we have,

$$\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + 37x - 5\frac{dx}{dt} = 0$$
$$\Rightarrow \frac{d^2x}{dt^2} - 12\frac{dx}{dt} + 37x = 0.$$

Similarly, as earlier, we get,

$$x = e^{6t}(c_1\cos t + c_2\sin t)$$

and $y = e^{6t}\{(c_1-c_2)\cos t + (c_1+c_2)\sin t\}$.

Example-2 :→ Solve $\frac{dx}{dt} + 5x + y = e^t$

$$\frac{dy}{dt} - x + 3y = e^{2t}.$$

Solution :→ The given equations are

$$(D+5)x + y = e^t \longrightarrow (1)$$
$$-x + (D+3)y = e^{2t} \longrightarrow (2)$$
, where $D \equiv \frac{d}{dt}$.

Eliminating y between (1) and (2), we have

$$(D+3)(D+5)x + (D+3)y - [-x + (D+3)y] = (D+3)e^t - e^{2t}$$
$$\Rightarrow (D^2+8D+15)x + x = e^t + 3e^t - e^{2t}$$
$$\Rightarrow (D^2+8D+16)x = 4e^t - e^{2t}.$$

The auxiliary equation is $m^2 + 8m + 16 = 0$

$$\Rightarrow (m+4)^2 = 0$$
$$\Rightarrow m = -4, -4$$

∴ The complementary function is

C.F. $= x_c = (c_1 + c_2 t)e^{-4t}$, where c_1 and c_2 are arbitrary constants.

The particular integral is

$$\text{P.I.} = x_p = \frac{1}{(D+4)^2}\cdot(4e^t - e^{2t})$$
$$= \frac{1}{(D+4)^2}\cdot 4e^t - \frac{1}{(D+4)^2}\cdot e^{2t}$$
$$= 4e^t \frac{1}{(1+4)^2} - e^{2t}\cdot\frac{1}{(2+4)^2}$$
$$= \frac{4}{25}e^t - \frac{1}{36}e^{2t}.$$

$$\therefore x = x_c + x_p = (c_1 + c_2 t)e^{-4t} + \frac{4}{25}e^t - \frac{1}{36}e^{2t} \longrightarrow (3)$$

$$\therefore \frac{dx}{dt} = c_2 e^{-4t} + (-4)(c_1 + c_2 t)e^{-4t} + \frac{4}{25}e^t - \frac{2}{36}e^{2t}$$

Putting the value of x and $\frac{dx}{dt}$ in (1) we have

$$y = e^t - \Big[(c_2 - 4c_1 - 4c_2 t)e^{-4t} + \frac{4}{25}e^t - \frac{1}{18}e^{2t} + 5(c_1 + c_2 t)e^{-4t} + \frac{4}{5}e^t - \frac{5}{36}e^{2t}\Big]$$
$$= -(c_1 + c_2 + c_2 t)e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t} \longrightarrow (4)$$

∴ Eqn (3) and (4) together constitute the general solution.

EXERCISE 3 →

1. Solve the following equations:

i) $\frac{dx}{dt} = 3x + 2y$

$\frac{dy}{dt} + 5x + 3y = 0$

ii) $\frac{dx}{dt} = -\omega y$

$\frac{dy}{dt} = \omega x$

iii) $\frac{dx}{dt} + y = e^t$

$\frac{dy}{dt} - x = e^{-t}$

iv) $-\frac{dx}{dt} + 3\frac{dy}{dt} + y = e^t$

$3\frac{dx}{dt} - \frac{dy}{dt} + x = 0$

v) $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \sin t$

$\frac{dx}{dt} + x - 3y = 0$

[Hint :→ From eqn ①, solving ① we have the value of y. Then putting that value in the eqn ②, we again solve eqn ② to get the value of x.]

Answers:

1-i) $x = c_1 \cos t + c_2 \sin t$, $y = \frac{1}{2}(c_2 - 3c_1)\cos t - \frac{1}{2}(c_1 + 3c_2)\sin t$

ii) $x = c_1 \cos\omega t + c_2 \sin\omega t$, $y = c_1 \sin\omega t - c_2 \cos\omega t$

iii) $x = (c_1 \cos t + c_2 \sin t) + \frac{1}{2}(e^t - e^{-t})$

$y = (c_1 \sin t \pm c_2 \cos t) + \frac{1}{2}(e^t - e^{-t})$

iv) $x = c_1 e^{-\frac{1}{2}t} + c_2 e^{-\frac{1}{4}t} + \frac{1}{15}e^t$, $y = c_1 e^{-\frac{1}{2}t} - c_2 e^{-\frac{1}{4}t} + \frac{4}{15}e^t$.

v) $x = \frac{3}{2}c_1 e^{2t} - 3c_2 e^{-t} - \frac{3}{10}(\cos t - 2\sin t)e^t$

$y = c_1 e^t + c_2 e^{-2t} - \frac{1}{10}(\cos t + 3\sin t)$.

To be continued

Prepared by
Pramita Bhar.